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Conservation laws and potential symmetries of systems of diffusion equations

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Abstract
We classify local first-order conservation laws for a class of systems of nonlinear diffusion equations. The derived conservation laws are used to construct the set of inequivalent potential systems for the class under consideration. Four potential systems are investigated from the Lie point of view and new potential symmetries are obtained. An example of the reduction of a system of diffusion equations with respect to a potential symmetry generator is given. A nonlinear system that has applications in plasma physics is linearized using infinite-dimensional potential symmetries.

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1. Introduction

The system of diffusion equations

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[f(u,v)\frac{\partial u}{\partial x} + p(u,v)\frac{\partial v}{\partial x}\right], \qquad \frac{\partial v}{\partial t} = \frac{\partial}{\partial x}\left[q(u,v)\frac{\partial u}{\partial x} + g(u,v)\frac{\partial v}{\partial x}\right] \tag{1}$$

contains a number of physically important models, in particular, an extension of Richards' equation, which describes the movement of water in a homogeneous unsaturated soil, to the system describing the combined transport of water vapor and heat under a combination of gradients of soil temperature and volumetric water content. Such coupled transport is of considerable significance in semi-arid environments where moisture transport often occurs essentially in the water vapor phase [9, 14].

In the present paper, we study local conservation laws and potential symmetries of the class of diffusion equations of the form

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[f(u,v)\frac{\partial u}{\partial x}\right], \qquad \frac{\partial v}{\partial t} = \frac{\partial}{\partial x}\left[g(u,v)\frac{\partial v}{\partial x}\right], \tag{2}$$

where f and g are arbitrary non-zero smooth functions in their arguments and in addition, $f_v^2 + g_u^2 \neq 0$. This class can be viewed as a special case of class (1). Furthermore special cases of this class of equations have been used to model successfully physical situations, such as transport in porous media with variable transmissivity [5] and river pollution [11]. Special cases of (2) also have applications in plasma physics [17, 18].

We recall a heuristic definition of conservation laws of a system of differential equations [12]. A *conservation law* of a system of PDEs $\mathcal{L}(x, u_{(r)}) = 0$ is a divergence expression $\operatorname{div} F = 0$ which vanishes for all solutions of this system. Here $x = (x_1, \ldots, x_n), u = (u^1, \ldots, u^m).F = (F^1, \ldots, F^n)$, where $F^i = F^i(x, u_{(r)})$, is a conserved vector of this conservation law, $u_{(r)}$ is the set of all partial derivatives of function u with respect to x of order not greater than r, including u as the derivative of zero order. The *order* of the conserved vector F is the maximal order of derivatives that explicitly appear in F.

In the framework of such non-rigorous definition, one can introduce the notion of triviality of conservation laws.

A conserved vector F is called *trivial* if $F^i = \hat{F}^i + \check{F}^i, i = \overline{1, n}$, where $\hat{F}^i$ and $\check{F}^i$ are, likewise F^i, functions of x and derivatives of u (i.e. differential functions), $\hat{F}^i$ vanish on the solutions of $\mathcal{L}$ and the n-tuple $\check{F} = (\check{F}^1, \ldots, \check{F}^n)$ is a null divergence (i.e. its divergence vanishes identically).

In a similar manner, a conservation law is trivial, if its conserved vector is trivial. Two conservation laws are equivalent, if their difference is a trivial conservation law. Conservation laws are called linearly dependent if there exists a linear combination of them which is a trivial conservation law.

Although the above definitions are suitable for the first intuitive illustration of the notion of conservation laws, in order to obtain complete and correct understanding we should use a more rigorous definition of conservation laws (see, e.g., [16, 22]). Namely, for any system $\mathcal{L}$ of differential equations the set $\mathrm{CV}(\mathcal{L})$ of its conserved vectors is a linear space, and the subset $\mathrm{CV}_0(\mathcal{L})$ of trivial conserved vectors is a linear subspace in $\mathrm{CV}(\mathcal{L})$. The factor space $\mathrm{CL}(\mathcal{L}) = \mathrm{CV}(\mathcal{L})/\,\mathrm{CV}_0(\mathcal{L})$ coincides with the set of equivalence classes of $\mathrm{CV}(\mathcal{L})$ with respect to the equivalence relation of conserved vectors.

Definition 1. *The elements of* $\mathrm{CL}(\mathcal{L})$ *are called conservation laws of the system* $\mathcal{L}$*, and the whole factor space* $\mathrm{CL}(\mathcal{L})$ *is called the space of conservation laws of* $\mathcal{L}$*.*

That is why we assume the description of the set of conservation laws as finding $\mathrm{CL}(\mathcal{L})$ that is equivalent to construction of either a basis if $\dim \mathrm{CL}(\mathcal{L}) < \infty$ or a system of generatrices in the infinite-dimensional case. The elements of $\mathrm{CV}(\mathcal{L})$ that belong to the same equivalence class giving a conservation law $\mathcal{F}$ are all considered as conserved vectors of this conservation law, and we will additionally identify elements from $\mathrm{CL}(\mathcal{L})$ with their representatives in $\mathrm{CV}(\mathcal{L})$. For $F \in \mathrm{CV}(\mathcal{L})$ and $\mathcal{F} \in \mathrm{CL}(\mathcal{L})$ the notation $F \in \mathcal{F}$ will denote that F is a conserved vector corresponding to the conservation law $\mathcal{F}$. In contrast to the order r_F of a conserved vector F as the maximal order of derivatives explicitly appearing in F, the *order of the conservation law* $\mathcal{F}$ is called $\min\{r_F | F \in \mathcal{F}\}$. Under linear dependence of conservation laws we understand linear dependence of them as elements of $\mathrm{CL}(\mathcal{L})$. Therefore, in the framework of the 'representative' approach conservation laws of a system $\mathcal{L}$ are considered *linearly dependent* if there exists a linear combination of their representatives, which is a trivial conserved vector.

Note 1. If a local transformation connects two systems of PDEs then under the action of this transformation a conservation law of the first of these systems is transformed into a conservation law of the second system, i.e. the equivalence transformation establishes a one-to-one correspondence between conservation laws of these systems. So, we can consider a problem of classification of conservation laws with respect to the equivalence group of the initial class. (See [16] for more details and rigorous definitions and proofs.)

The conservation laws are closely connected with nonlocal (potential) symmetries. A system of PDEs may admit symmetries of such sort when at least one of its equations can be written in a conserved form. After introducing potentials for PDEs written in the conserved form as additional dependent variables, we obtain a new (potential) system of PDEs. Any local invariance transformation of the obtained system induces a symmetry of the initial system. If transformations of some of the 'non-potential' variables explicitly depend on potentials, this symmetry is a nonlocal (potential) symmetry of the initial system, otherwise project into point symmetries of the initial system. More details about potential symmetries and their applications can be found in [3, 4]. This procedure of finding potential symmetries has been generalized in [16] by admitting dependence of symmetries and conservation laws on several potentials simultaneously. The most general method of choosing conservation laws for introducing potentials, in order to obtain all possible inequivalent potential systems, has been proposed recently in [7]. Different types of nonlocal symmetries were studied by other authors [1, 10].

Taking into account note 1, we will classify conservation laws for class (2) up to the group $G^{\sim}$ containing equivalence transformations

$$\tilde{t} = \varepsilon_5 t + \varepsilon_1, \qquad \tilde{x} = \varepsilon_6 x + \varepsilon_2, \qquad \tilde{u} = \varepsilon_7 u + \varepsilon_3,$$
$$\tilde{v} = \varepsilon_8 v + \varepsilon_4, \qquad \tilde{f} = \varepsilon_5^{-1}\varepsilon_6^2 f, \qquad \tilde{g} = \varepsilon_5^{-1}\varepsilon_6^2 g,$$

and a subgroup of discrete transformations $(u, v, f, g) \to (v, u, g, f)$. Here ε_i are arbitrary constants and in addition, $\varepsilon_5\varepsilon_6\varepsilon_7\varepsilon_8 \neq 0$.

The spaces of conservation laws of systems of form (2) are very rich (see the following section). Therefore, for class (2) there exists a large number of inequivalent potential systems. Some of these systems were investigated before [19–21]. Let us note that in [19] only preliminary analysis of potential symmetries associated with the 'usual' potential system (4) (see subsection 4.2) was presented. Here we complete this investigation and adduce the complete group classification of the potential system as well as classification of some other potential systems.

The paper is organized as follows. In section 2, we classify first-order local conservation laws of systems (2). Using this classification, in section 3, we adduce a list of inequivalent potential systems for systems from class (2). Then, in section 4, we classify potential symmetries of (2) associated with four of the inequivalent potential systems. A brief discussion of obtaining non-Lie exact solutions being invariant with respect to potential symmetry generators is given in section 5 together with an example of reduction of system (2) with respect to its potential symmetry. An example of an application of the obtained results to linearization of the model arising in plasma physics is presented in section 6.

2. Conservation laws

Since system (2) is $(1+1)$-dimensional, we search its conservation laws in the form

$$D_t T(t, x, u, v, u_t, u_x, v_t, v_x, \ldots) + DX(t, x, u, v, u_t, u_x, v_t, v_x, \ldots) = 0.$$

Here T is the conserved density and X is the conserved flux of the conservation law.

In the case of a single $(1+1)$-dimensional second-order evolution equation all conserved vectors are equivalent to the first-order ones [6, 16]. However, it is not true for the case of a system of diffusion equations. Thus, e.g., it is well known that the Schrödinger equation (that can be represented as a pair of equations of form (2) with $f = 1$, $g = -1$) possesses an infinite series of higher-order conservation laws.

Here we restrict ourselves by studying first-order conservation laws of system (2) having the form

$$D_t T(t, x, u, v, u_x, v_x) + DX(t, x, u, v, u_x, v_x) = 0.$$

For the construction of conservation laws we use modification of the direct method proposed in [16]. More precisely, we consider the above expression on the solution manifold of system (2), split it with respect to the unconstrained derivatives of functions u and v and solve the obtained determining equations up to the equivalence relation of conservation laws. As a result we have the following theorem.

Theorem 1. *A list of inequivalent first-order conserved vectors (T, X) of conservation laws of systems from class (2) is exhausted by the following ones.*

1. $\forall f, \forall g: \quad (u, -fu_x),\ (v, -gv_x)$.
2. $\forall f, g_u = f_v: \quad (u, -fu_x),\ (v, -gv_x), (xu + xv, -xfu_x - xgv_x + \int f\,\mathrm{d}u)$.
3. $\forall f, g = -f: \quad (u, -fu_x),\ (v, -gv_x),\ (uv, -fu_xv + fuv_x)$.
4. $\forall f, g = g(v): \quad (u, -fu_x),\ (v, -gv_x),\ (xv, -xgv_x + \int g dv)$.
5. $\forall g, f = 1: \quad (v, -gv_x),\ (\alpha u, -\alpha u_x + \alpha_x u)$, *where* $\alpha = \alpha(t, x)$ *runs through the solution set of the linear backward heat equation* $\alpha_t + \alpha_{xx} = 0$.
6. $f = -g = f(u - \varepsilon v), \varepsilon = 0, 1: \quad (u, -fu_x),\ (v, -gv_x),\ (uv, f(-u_xv + uv_x))$, $(xu - \varepsilon xv, -xf(u_x - \varepsilon v_x) + \int f\,\mathrm{d}u)$.
7. $f = -g = u^{-2}\varphi_\omega(\omega), \omega = vu^{-1}: \quad (u, -fu_x),\ (v, -gv_x),\ (uv, f(-u_xv + uv_x))$, $(xuv, -[\varphi + xf(u_xv - uv_x)])$.
8. $f = -g = \frac{1}{(u-\varepsilon v)^2}, \varepsilon = 0, 1: \quad (u, -fu_x),\ (v, -gv_x),\ (uv, -fu_xv + fuv_x)$, $(xu - \varepsilon xv, -xf(u_x - \varepsilon v_x) + \int f\,\mathrm{d}u)$, $\left(xuv, \frac{-xu_xv+xuv_x}{(u-\varepsilon v)^2} - \frac{u}{u-\varepsilon v}\right)$.

Note 2. The problem of investigation of higher-order conservation laws of systems (2) remains open.

3. Potential systems

The above set of conservation laws can be used for deriving potential systems for systems from class (2). Applying the technique proposed in [7] which is based on investigation of equivalences of pairs 'system+conservation law' with respect to groups of equivalence and symmetry transformations, we can formulate the following theorem.

Theorem 2. *A list of inequivalent (with respect to $G^\sim$ and symmetry transformations of the corresponding systems from class (2) potential systems for systems from class (2) is exhausted by the following ones:*

$\forall f, g$

1. $u_t = (fu_x)_x,\ w_x = v,\ w_t = gv_x$;
2. $u_t = (fu_x)_x,\ w_x = u + v,\ w_t = fu_x + gv_x$;
3. $w_x = u,\ w_t = fu_x,\ z_x = v,\ z_t = gv_x$;

$\forall f,\, g_u = f_v$

1.; *2.*; *3.*;

4. $u_t = (fu_x)_x,\, w_x = xu + xv + \alpha v,\, w_t = xfu_x + xgv_x - \int f\,\mathrm{d}u + \alpha g v_x$;
5. $w_x = xu + xv + \alpha v,\, w_t = xfu_x + xgv_x - \int f\,\mathrm{d}u + \alpha g v_x,\, z_x = v,\, z_t = gv_x$;
6. $w_x = xu + xv + \alpha v,\, w_t = xfu_x + xgv_x - \int f\,\mathrm{d}u + \alpha g v_x,\, z_x = u + \delta v,$
   $z_t = fu_x + \delta g v_x$;
7. $w_x = u,\, w_t = fu_x,\, z_x = v,\, z_t = gv_x,\, q_x = xu + xv,\, q_t = xfu_x + xgv_x - \int f\,\mathrm{d}u$;

$\forall f,\, g = -f$

1.; *2.*; *3.*;

8. $u_t = (fu_x)_x,\, w_x = uv,\, w_t = fu_x v - fuv_x$;
9. $w_x = uv,\, w_t = fu_x v - fuv_x,\, z_x = v,\, z_t = -fv_x$;
10. $w_x = uv,\, w_t = fu_x v - fuv_x,\, z_x = u + v,\, z_t = fu_x - fv_x$;
11. $w_x = u,\, w_t = fu_x,\, z_x = v,\, z_t = gv_x,\, q_x = uv,\, q_t = fu_x v - fuv_x$

$\forall f,\, g_u = 0$

1.; *2.*; *3.*;

12. $u_t = (fu_x)_x,\, w_x = xv + \delta u,\, w_t = xgv_x - \int g dv + \delta f u_x$;
13. $u_t = (fu_x)_x,\, w_x = v + \delta u,\, w_t = gv_x + \delta f u_x$;
14. $w_x = xv + \delta_1 u,\, w_t = xgv_x - \int g dv + \delta_1 f u_x,\, z_x = u,\, z_t = fu_x$;
15. $w_x = xv + \delta_1 u,\, w_t = xgv_x - \int g dv + \delta_1 f u_x,\, z_x = v + \delta_2 u,\, z_t = gv_x + \delta_2 f u_x$;
16. $w_x = u,\, w_t = fu_x,\, z_x = v,\, z_t = gv_x,\, q_x = xv,\, q_t = xgv_x - \int g dv$;

$\forall g,\, f = 1$

1.;

17. $w^i_x = \alpha^i u + \delta_i v,\, w^i_t = \alpha^i u_x - \alpha^i_x u + \delta_i g v_x,\, i = 1, \ldots, n,\, v_t = (gv_x)_x$, *where* $\alpha^1, \ldots,$ α^n *is a set of* n *linearly independent solutions of the linear backward heat equation* $\alpha_t + \alpha_{xx} = 0,\, n \geqslant 1$.

$f = -g = f(u - \varepsilon v)$

1.; *2.*; *3.*; *8.*; *9.*; *10.*; *11.*;

18. $u_t = (fu_x)_x,\, w_x = xu - \varepsilon xv + \delta_1 uv + c_2 v,\, w_t = xf(u_x - \varepsilon v_x) - \int f\,\mathrm{d}u$
    $+ \delta_1 f(u_x v - uv_x) - \delta_2 f v_x$;
19. $w_x = xu - \varepsilon xv + \delta_1 uv + \delta_2 v,\, w_t = xf(u_x - \varepsilon v_x) - \int f\,\mathrm{d}u + \delta_1 f(u_x v - uv_x)$
    $- \delta_2 f v_x,\, z_x = \delta_3 uv + \delta_4 u + \delta_5 v,\, z_t = \delta_3 f(u_x v - uv_x) + \delta_4 f u_x - \delta_5 f v_x,$
    where $1 \leqslant \delta_3^2 + \delta_4^2 + \delta_5^2 \leqslant 2$;
20. $w_x = xu - \varepsilon xv + \delta_1 v,\, w_t = xf(u_x - \varepsilon v_x) - \int f\,\mathrm{d}u - \delta_1 f v_x,\, z_x = uv,$
    $z_t = f(u_x v - uv_x),\, q_x = u + \delta_2 v,\, q_t = fu_x - fv_x$, *where* $\delta_1\delta_2 = 0$;
21. $w_x = xu - \varepsilon xv + \delta_1 uv,\, w_t = xf(u_x - \varepsilon v_x) - \int f\,\mathrm{d}u + \delta_1 f(u_x v - uv_x),$
    $z_x = u,\, z_t = fu_x,\, q_x = v,\, q_t = -fv_x$;

22. $w_x = u, w_t = fu_x, z_x = v, z_t = gv_x, q_x = uv, q_t = f(u_xv - uv_x),$
$r_x = xu - \varepsilon xv, r_t = xf(u_x - \varepsilon v_x) - \int f\,\mathrm{d}u;$

$$f = -g = u^{-2}\varphi_\omega(\omega), \omega = vu^{-1}$$

1.; *2.*; *3.*; *8.*; *9.*; *10.*; *11.*;

23. $u_t = (fu_x)_x, w_x = xuv + \delta_1 u + \delta_2 v, w_t = \varphi + xf(u_xv - uv_x) + \delta_1 fu_x - \delta_2 fv_x;$

24. $w_x = xuv + \delta_1 u + \delta_2 v, w_t = \varphi + xf(u_xv - uv_x) + \delta_1 fu_x - \delta_2 fv_x,$
$z_x = \delta_3 uv + \delta_4 u + \delta_5 v, z_t = \delta_3 f(u_xv - uv_x) + \delta_4 fu_x - \delta_5 fv_x,$
where $\delta_3^2 + \delta_4^2 + \delta_5^2 \neq 0, \delta_3\delta_4\delta_5 = 0, \delta_1\delta_4 = 0, \delta_2\delta_5 = 0;$

25. $w_x = xuv + \delta_1 u + \delta_2 v, w_t = \varphi + xf(u_xv - uv_x) + \delta_1 fu_x - \delta_2 fv_x,$
$z_x = uv, z_t = f(u_xv - uv_x), q_x = \delta_3 u + \delta_4 v, q_t = \delta_3 fu_x - \delta_4 fv_x,$
where $\delta_3^2 + \delta_4^2 = 1, \delta_1\delta_3 = 0, \delta_2\delta_4 = 0;$

26. $w_x = xuv, w_t = \varphi + xf(u_xv - uv_x), z_x = u, z_t = fu_x, q_x = v, q_t = -fv_x;$

27. $w_x = u, w_t = fu_x, z_x = v, z_t = gv_x, q_x = uv, q_t = f(u_xv - uv_x),$
$r_x = xuv, r_t = \varphi + xf(u_xv - uv_x);$

$$\forall f = -g = (u - \varepsilon v)^{-2}$$

1.; *2.*; *3.*; *8.*; *9.*; *10.*; *11.*; *20.–27.*;

28. $w_x = xuv + \delta_1 u + \delta_2 v, w_t = \frac{u}{u-\varepsilon v} + (xu_xv - xuv_x)f + \delta_1 fu_x - \delta_2 fv_x,$
$z_x = xu - \varepsilon xv + \delta_3 uv + \delta_4 u + \delta_5 v, z_t = xf(u_x - \varepsilon v_x) - \int f\,\mathrm{d}u + \delta_3 f(u_xv - uv_x)$
$+ \delta_4 fu_x - \delta_5 fv_x;$

29. $w_x = xuv + \delta_1 u + \delta_2 v, w_t = \frac{u}{u-\varepsilon v} + (xu_xv - xuv_x)f + \delta_1 fu_x - \delta_2 fv_x,$
$z_x = xu - \varepsilon xv + \delta_3 uv + \delta_4 u + \delta_5 v, z_t = xf(u_x - \varepsilon v_x) - \int f\,\mathrm{d}u$
$+ \delta_3 f(u_xv - uv_x) + \delta_4 fu_x - \delta_5 fv_x, q_x = \delta_6 uv + \delta_7 u + \delta_8 v, q_t = \delta_6 f(u_xv - uv_x)$
$+ \delta_7 fu_x - \delta_8 fv_x;$

30. $w_x = xuv + \delta_1 u + \delta_2 v, w_t = \frac{u}{u-\varepsilon v} + (xu_xv - xuv_x)f + \delta_1 fu_x - \delta_2 fv_x,$
$z_x = xu - \varepsilon xv + \delta_3 u + \delta_4 v, z_t = xf(u_x - \varepsilon v_x) - \int f\,\mathrm{d}u + \delta_3 fu_x - \delta_4 fv_x,$
$q_x = uv + \delta_5 u + \delta_6 v, q_t = f(u_xv - uv_x) + \delta_5 fu_x - \delta_6 fv_x, r_x = \delta_7 u + \delta_8 v,$
$r_t = \delta_7 fu_x - \delta_8 fv_x$, *where* $\delta_7^2 + \delta_8^2 \neq 0, \delta_7\delta_5 = \delta_7\delta_1 = \delta_8\delta_6 = \delta_8\delta_2 = 0;$

31. $w_x = xuv, w_t = \frac{u}{u-\varepsilon v} + (xu_xv - xuv_x)f, z_x = xu - \varepsilon xv + \delta_3 uv,$
$z_t = xf(u_x - \varepsilon v_x) - \int f\,\mathrm{d}u + \delta_3 f(u_xv - uv_x), q_x = u, q_t = fu_x, r_x = v, r_t = gv_x,$

32. $w_x = u, w_t = fu_x, z_x = v, z_t = gv_x, q_x = uv, q_t = f(u_xv - uv_x), r_x = xu - \varepsilon xv,$
$r_t = xf(u_x - \varepsilon v_x) - \int f\,\mathrm{d}u, p_x = xuv, p_t = \frac{u}{u-\varepsilon v} + (xu_xv - xuv_x)f.$

Here $\alpha, \delta, \delta_i = \mathrm{const}, (\alpha, \delta) \neq (0, 0).$

Note 3. In some of the cases we can also obtain additional restrictions on values of parameters δ_i using the scaling equivalence transformations.

As one can see, there exist three inequivalent potential systems that are common for all values of the parameter functions f and g. Potential symmetries of (2) associated with these systems and also to the potential system (8) will be investigated in the following section.

4. Potential symmetries

It is known [16] that the equivalence group for a class of systems or the symmetry group for single system can be prolonged to potential variables. It is natural to use these prolonged equivalence groups for classification of possible potential symmetries. In view of this statement we will classify potential symmetries of class (2) up to the (trivial) prolongation of group $G^{\sim}$ to the corresponding potentials.

4.1. Classification of common potential systems with one potential

We search for potential symmetries of system (2) associated with the potential system

$$w_x = u, \qquad w_t = f u_x, \qquad v_t = (g v_x)_x. \tag{3}$$

That is, we look for Lie symmetry generators for (3) of the form

$$Q = \tau(t,x,u,v,w)\partial_t + \xi(t,x,u,v,w)\partial_x + \eta(t,x,u,v,w)\partial_u + \zeta(t,x,u,v,w)\partial_v + \kappa(t,x,u,v,w)\partial_w.$$

System (3) is a special case of a general system considered in [20]. There exist nine inequivalent systems of form (3) leading to potential symmetries of systems (2) which are tabulated in the list below. Cases 1–7 appear in [20] and cases 8 and 9 are new. In each case we give the functional forms of f and g and the basis generators of the potential symmetry algebra.

(1) $f = (u^2+q)^{-1}\,\mathrm{e}^{\int r(u^2+q)^{-1}\mathrm{d}u}\varphi(\omega)$, $\omega = v\,\mathrm{e}^{\int r(u^2+q)^{-1}\mathrm{d}u}$, $g = -f$ $\langle \partial_t, \partial_x, \partial_w, 2t\partial_t + x\partial_x + w\partial_w, rt\partial_t + w\partial_x - (u^2+q)\partial_u + rv\partial_v - qx\partial_w\rangle$;

(2) $f = (u^2+q)^{-1}\,\mathrm{e}^{\int r(u^2+q)^{-1}\mathrm{d}u}\varphi(\omega)$, $\omega = v + \int r(u^2+q)^{-1}\mathrm{d}u$, $g = -f$ $\langle \partial_t, \partial_x, \partial_w, 2t\partial_t + x\partial_x + w\partial_w, rt\partial_t + w\partial_x - (u^2+q)\partial_u + r\partial_v - qx\partial_w\rangle$;

(3) $f = u^{-2}v^{-4/3}$, $g = -f$ $\langle \partial_t, \partial_x, \partial_w, 2t\partial_t + x\partial_x + w\partial_w, 2t\partial_t + x\partial_x + v\partial_v, x\partial_x - u\partial_u, w\partial_x - u^2\partial_u, xw\partial_x + u(w-xu)\partial_u - 3vw\partial_v + w^2\partial_w\rangle$;

(4) $f = u^{-2}$, $g = -f$ $\langle \partial_t, \partial_w, 2t\partial_t + x\partial_x + w\partial_w, x\partial_x - u\partial_u, v\partial_v, xw\partial_x - u(w+xu)\partial_u - vw\partial_v - 2t\partial_w, 4t^2\partial_t - x(w^2+2t)\partial_x + u(6t+2xuw+w^2)\partial_u + v(w^2-2t)\partial_v + 4tw\partial_w, \alpha\partial_x - u^2\alpha_w\partial_u, \beta\partial_v\rangle$, where the function $\alpha = \alpha(t,w)$ satisfies the linear heat equation $\alpha_t - \alpha_{ww} = 0$ and $\beta = \beta(t,w)$ satisfies the backward linear heat equation $\beta_t + \beta_{ww} = 0$;

(5) $f = (u^2+q)^{-1}\mathrm{e}^{\int r(u^2+q)^{-1}\mathrm{d}u}$, $g = -f$ $\langle \partial_t, \partial_x, \partial_v, \partial_w, 2t\partial_t + x\partial_x + w\partial_w, v\partial_v, (-rx+2w)\partial_x - 2(u^2+q)\partial_u - 2v\partial_v - (2qx+rw)\partial_w\rangle$;

(6) $f = ug_u + g$, $g = g(u)$ $\langle \partial_t, \partial_x, \partial_v, \partial_w, 2t\partial_t + x\partial_x + w\partial_w, v\partial_v, w\partial_v\rangle$;

(7) $f = v^{-4/3}$, $g = \mu v^{-4/3}$ $\langle \partial_t, \partial_x, \partial_w, 2t\partial_t + x\partial_x + w\partial_w, \partial_u + x\partial_w, 2t\partial_t + x\partial_x - u\partial_u, 4t\partial_t + 3v\partial_v, x^2\partial_x + (w-xu)\partial_u - 3xv\partial_v + xw\partial_w\rangle$;

(8) $f = v^n u^{-2}$, $g = -f$ $\langle \partial_t, \partial_x, \partial_v, \partial_w, 2t\partial_t + x\partial_x + w\partial_w, -nt\partial_t + v\partial_v, x\partial_x - u\partial_u, w\partial_x - u^2\partial_u\rangle$;

(9) $f = (uv+1)^{-2}$, $g = -f$ $\langle \partial_t, \partial_x, \partial_v, \partial_w, 2t\partial_t + x\partial_x + w\partial_w, u\partial_u - v\partial_v + w\partial_w, w\partial_x - u^2\partial_u - \partial_v\rangle$.

We point out that symmetry generators that satisfy the condition

$$\tau_w^2 + \xi_w^2 + \eta_w^2 + \zeta_w^2 \neq 0$$

are potential symmetries, while the rest project into Lie point symmetries for the original system (2).

As was shown in the previous section, for system (2) with arbitrary values of elements f and g there exist one more inequivalent potential system associated with one potential, namely

$$u_t = (fu_x)_x, \qquad w_x = u + v, \qquad w_t = fu_x + gv_x.$$

It is not difficult to show that the Lie symmetries of this system do not imply potential symmetries for systems from class (2). That is, all Lie symmetries of this system project into Lie point symmetries for the original system (2).

4.2. *Classification of common potential system with two potentials*

Another potential system common for all values of arbitrary elements f and g is

$$w_x = u, \qquad w_t = fu_x, \qquad z_x = v, \qquad z_t = gv_x. \tag{4}$$

The potential symmetries derived in this subsection complete the results of [19]. We look for symmetry generator of system (4) of the form

$$Q = \tau(t,x,u,v,w,z)\partial_t + \xi(t,x,u,v,w,z)\partial_x + \eta(t,x,u,v,w,z)\partial_u + \zeta(t,x,u,v,w,z)\partial_v + \kappa(t,x,u,v,w,z)\partial_w + \lambda(t,x,u,v,w,z)\partial_z.$$

In order to obtain potential symmetries of systems (2) we have to impose condition that at least one of the functions τ, ξ, η, ζ depends explicitly on potential variables w or/and z.

If $f \neq g$ then there is no potential symmetries. This agrees with [19]. We therefore consider the case $f = g$.

Using the classical Lie infinitesimal approach we construct determining equations for coefficients of the potential symmetry generator:

$$\tau_x = \tau_u = \tau_v = \tau_w = \tau_z = 0, \qquad \xi_u = \kappa_u = \lambda_u = \xi_v = \kappa_v = \lambda_v = 0,$$
$$\eta = -\xi_w u^2 + (\kappa_w - \xi_x)u - \xi_z uv + \kappa_z v + \kappa_x,$$
$$\zeta = -\xi_z v^2 + (\lambda_z - \xi_x)v - \xi_w uv + \lambda_w u + \lambda_x,$$
$$f(\eta_w u + \eta_z v + \eta_x) + \xi_t u - \kappa_t = 0,$$
$$f(\zeta_z v + \zeta_w u + \zeta_x) + \xi_t v - \lambda_t = 0,$$
$$\eta f_u + \zeta f_v + (-2\xi_w u - 2\xi_z v - 2\xi_x + \tau_t)f = 0.$$

The last determining equation has the form

$$(Au^2 + Buv + au + bv + c)f_u + (Bv^2 + Auv + mu + nv + l)f_v + (2Au + 2Bv + r)f = 0$$

with respect to f, where A, B, a, b, c, m, n, l and r are constants. The number k of such independent equations can be equal to 0, 1, 2 or 3. If $k > 3$ then the system for f is incompatible. (More details and references about 'compatibility method' see, e.g., in [8, 15].)

If $k = 0$ then $\xi_w = \xi_z = \eta = \zeta = 0$. Therefore, there are no potential symmetry in this case.

If $k = 1$ then analyzing the last three determining equations we obtain, in particular,

$$\eta_w = \eta_z = \zeta_w = \zeta_z = \xi_{ww} = \xi_{zw} = \xi_{zz} = \kappa_{zz} = \kappa_{wz} = \lambda_{ww} = \lambda_{wz} = 0.$$

Therefore, potential symmetries exist iff $(A, B) \neq (0, 0)$. Using equivalence transformations, we can put $A = 1, r = 0$ and get the first case of potential symmetry for the initial system.

1. f is a solution of equation

$$(u^2 + Buv + au + bv + c)f_u + (Bv^2 + uv + mu + nv + l)f_v + (2u + 2Bv)f = 0.$$

Potential symmetry algebra

$$\begin{aligned}&\langle \partial_t, \partial_x, \partial_w, \partial_z, 2t\partial_t + x\partial_x + w\partial_w + z\partial_z, (w + Bz)\partial_x\\&\qquad + (-u^2 - Buv + au + bv + c)\partial_u + (-Bv^2 - uv + nv + mu + l)\partial_v\\&\qquad + (aw + bz + cx)\partial_w + (mw + nz + lx)\partial_z\rangle.\end{aligned}$$

If $k = 2$ then we have the following cases:

2. f is a solution of the system

$$\begin{aligned}&(u^2 + a_1u + b_1v + c_1)f_u + (uv + m_1u + n_1v + l_1)f_v + 2uf = 0,\\&(uv + a_2u + b_2v + c_2)f_u + (v^2 + m_2u + n_2v + l_2)f_v + 2vf = 0.\end{aligned}$$

Potential symmetry algebra

$$\begin{aligned}&\langle \partial_t, \partial_x, \partial_w, \partial_z, 2t\partial_t + x\partial_x + w\partial_w + z\partial_z, w\partial_x\\&\qquad + (-u^2 + a_1u + b_1v + c_1)\partial_u + (-uv + n_1v + m_1u + l_1)\partial_v\\&\qquad + (a_1w + b_1z + c_1x)\partial_w + (m_1w + n_1z + l_1x)\partial_z, z\partial_x\\&\qquad + (-uv + a_2u + b_2v + c_2)\partial_u + (-v^2 + n_2v + m_2u + l_2)\partial_v\\&\qquad + (a_2w + b_2z + c_2x)\partial_w + (m_2w + n_2z + l_2x)\partial_z\rangle.\end{aligned}$$

3. f is a solution of the system

$$\begin{aligned}&(u^2 + B_1uv + a_1u + b_1v + c_1)f_u + (B_1v^2 + uv + m_1u + n_1v + l_1)f_v + (2u + 2B_1v)f = 0,\\&(a_2u + b_2v + c_2)f_u + (m_2u + n_2v + l_2)f_v + r_2f = 0.\end{aligned}$$

Potential symmetry algebra

$$\begin{aligned}&\langle \partial_t, \partial_x, \partial_w, \partial_z, 2t\partial_t + x\partial_x + w\partial_w + z\partial_z,\\&(w + B_1z)\partial_x + (-u^2 - B_1uv + a_1u + b_1v + c_1)\partial_u + (-B_1v^2 - uv + n_1v + m_1u + l_1)\partial_v\\&\qquad + (a_1w + b_1z + c_1x)\partial_w + (m_1w + n_1z + l_1x)\partial_z,\\&r_2t\partial_t + (a_2u + b_2v + c_2)\partial_u + (n_2v + m_2u + l_2)\partial_v + (a_2w + b_2z + c_2x)\partial_w\\&\qquad + (m_2w + n_2z + l_2x)\partial_z\rangle.\end{aligned}$$

Note 4. Both of the above systems for f are compatible.

The last and the most interesting case is $k = 3$. It can be proved that up to $G^\sim$, the function f solves a system of the form

$$\begin{aligned}&(u^2 + B_1uv + a_1u + b_1v + c_1)f_u + (B_1v^2 + uv + m_1u + n_1v + l_1)f_v\\&\qquad + (2u + 2B_1v + r_1)f = 0,\\&(a_2u + b_2v + c_2)f_u + (m_2u + n_2v + l_2)f_v + r_2f = 0,\\&(a_3u + b_3v + c_3)f_u + (m_3u + n_3v + l_3)f_v + r_3f = 0,\end{aligned}$$

where the vectors (a_2, b_2, c_2) and (a_3, b_3, c_3) are not collinear.

All possible $G^\sim$ inequivalent solutions of this system compatible with the remaining classifying equations are $f = 1$ and those listed below:

4. $f = (u^2 + \varepsilon v^2 \pm 1)^{-1}, \varepsilon = \pm 1$

$$\begin{aligned}&\langle \partial_t, \partial_x, \partial_w, \partial_z, 2t\partial_t + x\partial_x + w\partial_w + z\partial_z, u\partial_v - \varepsilon v\partial_u + w\partial_z - \varepsilon z\partial_w,\\&- w\partial_x + (u^2 \pm 1)\partial_u + uv\partial_v \pm x\partial_w, -\varepsilon z\partial_x + \varepsilon uv\partial_u + (\epsilon v^2 \pm 1)\partial_v \pm x\partial_z\rangle.\end{aligned}$$

5 $f = (u^2 + \varepsilon v^2)^{-1}, \varepsilon = \pm 1$

$$\langle \partial_t, \partial_x, \partial_w, \partial_z, 2t\partial_t + x\partial_x + w\partial_w + z\partial_z, x\partial_x - u\partial_u - v\partial_v, u\partial_v - \varepsilon v\partial_u + w\partial_z - \varepsilon z\partial_w,$$
$$- w\partial_x + u^2\partial_u + uv\partial_v, -z\partial_x + uv\partial_u + v^2\partial_v,$$
$$- (2t + w^2 + \varepsilon z^2)\partial_x + 2(uw + \varepsilon vz)(u\partial_u + v\partial_v)\rangle.$$

6. $f = (u^2 \pm 1)^{-1}$

$$\langle \partial_t, \partial_x, \partial_v + x\partial_z, \partial_w, \partial_z, 2t\partial_t + x\partial_x + w\partial_w + z\partial_z, v\partial_v + z\partial_z, u\partial_v + w\partial_z,$$
$$- w\partial_x + (u^2 \pm 1)\partial_u + uv\partial_v \pm x\partial_w, 2(uw \pm x)\partial_v + (2t + w^2 \pm x^2)\partial_z\rangle.$$

7. $f = u^{-2}$

$$\langle \partial_t, \partial_x, \partial_v + x\partial_w, 2t\partial_t + x\partial_x + w\partial_w + z\partial_z, v\partial_v + z\partial_z, x\partial_x - u\partial_u - v\partial_v,$$
$$z\partial_x - uv\partial_u - v^2\partial_v, -xw\partial_x + (uw + xu^2)\partial_u + (xuv - uz)\partial_v + 2t\partial_w - wz\partial_z,$$
$$4t^2\partial_t - (xw^2 + 2tx)\partial_x + (6tu + uw^2 + 2xu^2w)\partial_u$$
$$+ (2xuvw - 2uwz)\partial_v + 4tw\partial_w - (2tz + w^2z)\partial_z,$$
$$\alpha\partial_x - u^2\alpha_w\partial_u - uv\alpha_w\partial_v, \alpha\partial_z + u\alpha_w\partial_v\rangle,$$

where $\alpha = \alpha(t, w)$ is an arbitrary solution of the linear heat equation $\alpha_t = \alpha_{ww}$.

8. $f = (u + \delta v)^{-1}, \delta = 0, 1$

$$\langle \partial_t, \partial_x, \partial_w, \partial_z, 2t\partial_t + x\partial_x + w\partial_w + z\partial_z, t\partial_t + x\partial_x - u\partial_u - v\partial_v,$$
$$\delta(\partial_u + x\partial_w) - (\partial_v + x\partial_z), t\partial_t + (u + \delta v)\partial_u + (w + \delta z)\partial_w,$$
$$\delta u\partial_u - u\partial_v + \delta w\partial_w - w\partial_z,$$
$$(\delta z + \delta xv + w + xu)(\delta\partial_u - \partial_v) + (2t + xw + \delta xz)(\delta w - \partial_z)\rangle.$$

9. $f = ((u + v)^2 \pm 1)^{-1}$

$$\langle \partial_t, \partial_x, \partial_w, \partial_z, 2t\partial_t + x\partial_x + w\partial_w + z\partial_z, \partial_u - \partial_v + x(\partial_w - \partial_z),$$
$$u(\partial_u - \partial_v) + w(\partial_w - \partial_z), v(\partial_u - \partial_v) + z(\partial_w - \partial_z),$$
$$- (w + z)\partial_x + u(u + v)\partial_u + (v^2 + uv \pm 1)\partial_v \pm x\partial_z,$$
$$2(uz + vw + uw + vz \pm x)(\partial_u - \partial_v) + (2t + w^2 \pm x^2 + z^2 + 2wz)(\partial_w - \partial_z)\rangle.$$

10. $f = (u + v)^{-2}$

$$\langle \partial_t, \partial_v + x\partial_w, \partial_z, 2t\partial_t + x\partial_x + w\partial_w + z\partial_z, -v\partial_u + v\partial_v - z\partial_w + z\partial_z, x\partial_x - u\partial_u - v\partial_v,$$
$$z\partial_x - uv\partial_u - v^2\partial_v, -x(w + z)\partial_x + (u + v)(w + 2z + xu)\partial_u$$
$$+ (u + v)(xv - z)\partial_v + (2t + z(w + z))\partial_w - (w + z)z\partial_z,$$
$$4t^2\partial_t - (x(w + z)^2 + 2tx)\partial_x + (u + v)(6t + (w + z)^2 + 2x(w + z)u + 2z(w + z))\partial_u$$
$$+ 2(u + v)(w + z)(xv - z)\partial_v + (4tw + 6tz + (w + z)^2z)\partial_w - (2tz + (w + z)^2z)\partial_z,$$
$$\varphi\partial_x - (u + v)\varphi_\omega(u\partial_u + v\partial_v), \varphi(\partial_w - \partial_z) + (u + v)\varphi_\omega(\partial_u - \partial_v)\rangle,$$

where $\omega = w + z$ and $\varphi = \varphi(t, \omega)$ is an arbitrary solution of the linear heat equation $\varphi_t = \varphi_{\omega\omega}$.

11. $f = (u^2 + v)^{-1}$

$$\langle \partial_t, \partial_x, \partial_w, \partial_z, 2t\partial_t + x\partial_x + w\partial_w + z\partial_z, 2t\partial_t + u\partial_u + 2v\partial_v + w\partial_w + 2z\partial_z,$$
$$\partial_u - 2u\partial_v + x\partial_w - 2w\partial_z, -2w\partial_x + (2u^2 + v)\partial_u + 2uv\partial_v + z\partial_w \rangle.$$

12. $f = (uv + 1)^{-1}$

$$\langle \partial_t, \partial_x, \partial_w, \partial_z, 2t\partial_t + x\partial_x + w\partial_w + z\partial_z, u\partial_u - v\partial_v + w\partial_w - z\partial_z,$$
$$w\partial_x - u^2\partial_u - (uv + 2)\partial_v - 2x\partial_z, z\partial_x - (uv + 2)\partial_u - v^2\partial_v - 2x\partial_w \rangle.$$

13. $f = u^{-1}v^{-1}$

$$\langle \partial_t, \partial_x, \partial_w, \partial_z, 2t\partial_t + x\partial_x + w\partial_w + z\partial_z, u\partial_u - v\partial_v + w\partial_w - z\partial_z,$$
$$x\partial_x - u\partial_u - v\partial_v, w\partial_x - u^2\partial_u - uv\partial_v, z\partial_x - uv\partial_u - v^2\partial_v,$$
$$(2t + wz)\partial_x - (vw + uz)(u\partial_u + v\partial_v) \rangle.$$

Thus, we get complete classification of potential symmetries of system (2) associated with potential system (4).

4.3. Classification of a special potential system

Let us consider now the system

$$w_x = uv, \qquad w_t = fu_xv - fuv_x, \qquad u_t = (fu_x)_x,$$

which form a potential system for system (2) with special value of parameter function $g = -f$. Using the classical Lie infinitesimal method we prove that there exists only one form of the parameter function $f = u^{-2}v^{-2}$ for which system (2) admits potential symmetries. In this case, the Lie algebra of potential symmetries has the following form:

$$\langle \partial_t, \partial_x, \partial_w, x\partial_x - u\partial_u, u\partial_u - v\partial_v, 2t\partial_t + u\partial_u + w\partial_w, w(u\partial_u - v\partial_v) + 2t\partial_w,$$
$$4t^2\partial_t + (2t + w^2)u\partial_u + (2t - w^2)v\partial_v + 4tw\partial_w \rangle. \tag{5}$$

5. Reductions with respect to potential symmetry generators

Potential symmetries of a system of differential equations can be used, e.g., for construction of exact solutions of the system. Namely, invariance with respect to a one-parameter group of symmetries leads to possibility of reducing the number of independent variables by one. For a case of two-dimensional systems in such a way one obtains a reduced system of ordinary differential equations.

Consider a system

$$u_t = (u^{-2}v^{-2}u_x)_x, \qquad v_t = (u^{-2}v^{-2}v_x)_x,$$

admitting nontrivial potential symmetries found in subsection 4.3. Its Lie symmetry algebra is five-dimensional and has the form

$$A^{\text{Lie}} = \langle \partial_t, \partial_x, 2t\partial_t + x\partial_x, x\partial_x - u\partial_u, x\partial_x - v\partial_v \rangle,$$

while the potential symmetry algebra $A^{\rm pot}$ is eight-dimensional and spanned by

$$v_1 = \partial_t,\ v_2 = \partial_x,\ v_3 = \partial_w,\ v_4 = x\partial_x - u\partial_u,\ v_5 = u\partial_u - v\partial_v,\ v_6 = 2t\partial_t + u\partial_u + w\partial_w,$$
$$v_7 = w(u\partial_u - v\partial_v) + 2t\partial_w,\ v_8 = 4t^2\partial_t + (2t + w^2)u\partial_u + (2t - w^2)v\partial_v + 4tw\partial_w.$$

Any two conjugate subgroups of a Lie symmetry group of a system of differential equations give rise to reduced equations that are related by a conjugacy transformation in the point symmetry group of the system acting on the invariant solutions determined by each subgroup [13]. Hence, up to the action of the point symmetry transformations, all invariant solutions for a given system can be obtained by selecting a subgroup in each conjugacy class of all admitted one-dimensional point symmetry subgroups. Such a selection is called an optimal set of one-dimensional subgroups. A set of subalgebras of the Lie algebra corresponding to the optimal set of subgroups consists of subalgebras inequivalent with respect to the actions of adjoint representation of the Lie symmetry group on its Lie algebra.

Therefore, to obtain non-Lie invariant solutions being invariant with respect to potential symmetry generators we have to find subalgebras of $A^{\rm pot}$ being inequivalent with respect to the actions of adjoint representation of the Lie symmetry transformations on the whole potential symmetry algebra.

Using the method proposed in [13] (see also its clear and simple explanation in [12]) we classify one-dimensional subalgebras of $A^{\rm pot}$ inequivalent with respect to inner automorphisms generated by the Lie symmetry transformations:

$$\langle v_8 + a_5v_5 + \delta_4v_4 + \varepsilon_3v_3 + a_1v_1\rangle,\ \langle v_8 + a_5v_5 + \varepsilon_3v_3 + \varepsilon_2v_2 + a_1v_1\rangle,$$
$$\langle v_7 + a_6v_6 + \delta_4v_4 + \varepsilon_3v_3 + a_1v_1\rangle,\ \langle v_7 + a_6v_6 + \varepsilon_3v_3 + \varepsilon_2v_2 + a_1v_1\rangle,$$
$$\langle v_6 + a_5v_5 + \delta_4v_4\rangle,\ \langle v_6 + a_5v_5 + \varepsilon_2v_2\rangle,\ \langle v_4 + a_5v_5 + \varepsilon_3v_3 + a_1v_1\rangle,$$
$$\langle v_5 + \varepsilon_3v_3 + \varepsilon_2v_2 + a_1v_1\rangle,\ \langle v_3 + \varepsilon_2v_2 + \varepsilon_1v_1\rangle,\ \langle v_2 + \varepsilon_1v_1\rangle,\ \langle v_1\rangle.$$

Here $\varepsilon_i = 0, \pm1$, $\delta_i = \pm1$ and a_i are arbitrary real constants.

Basis operators of the fifth–ninth of them are projectible on the space of variables (t, x, u, v). Therefore, reductions with respect to them lead to Lie invariant solutions. Reductions with respect to the first four subalgebras lead to exact solutions that may be non-Lie. Acting on these solutions by the Lie symmetries one can construct all possible potentially invariant solutions that cannot be obtained from Lie symmetries.

For example, solutions invariant with respect to subalgebra $\langle v_8 + \varepsilon_2v_2\rangle$ have the following form:

$$u = t^{1/2}\theta(\xi)\,{\rm e}^{t\varphi^2(\xi)/4}, \qquad v = t^{1/2}\psi(\xi)\,{\rm e}^{-t\varphi^2(\xi)/4}, \qquad w = t\varphi(\xi),$$

where $\xi = x + \frac{\varepsilon_2}{4t}$ and functions ψ, θ and φ satisfy the following system of equations:

$$\varphi' = \theta\psi, \qquad \varepsilon_2(\varphi')^3 = 4(\theta\psi' - \theta'\psi), \qquad 2\theta\theta''\psi = 3\theta\theta'\psi' + 5(\theta')^2.$$

6. An example from plasma physics

The system of equations

$$\begin{aligned} \rho_t &= [T^\mu\rho^m\rho_x]_x \\ P_t &= [T^\nu\rho^{n+1}T_x]_x + [T^{\mu+1}\rho^m\rho_x]_x \end{aligned} \tag{6}$$

is used to study the asymptotic behavior of a plasma slowly diffusing across a strong magnetic field [18]. Here P is the plasma pressure, ρ is the density and T is the temperature. It is assumed that $P = \rho T$. In the case $m = n = \mu = \nu = -2$, system (6) takes the form

$$\rho_t = (T^{-2}\rho^{-2}\rho_x)_x, \qquad (\rho T)_t = (\rho^{-1}T^{-2}T_x)_x + (\rho^{-2}T^{-1}\rho_x)_x.$$

Equivalently it can be written in the form

$$\rho_t = (T^{-2}\rho^{-2}\rho_x)_x, \qquad T_t = \rho^{-2}(T^{-2}T_x)_x$$

or if we re-introduce the pressure $P = \rho T$, then the system takes the form

$$\rho_t = (P^{-2}\rho_x)_x, \qquad P_t = (P^{-2}P_x)_x \tag{7}$$

which is a member of the general class (2).

If a nonlinear PDE (or a system of nonlinear PDEs) admits infinite-parameter symmetry groups then the nonlinear PDE (or the system of nonlinear PDEs) can be transformed into a linear PDE (or into a system of linear PDEs) provided that these groups satisfy certain criteria [3]. Now in the spirit of [2], where the nonlinear equation $u_t = [u^{-2}u_x]_x$ was linearized, we derive a nonlocal mapping that linearizes system (7). System (7) does not admit an infinite-parameter group. However, introducing the potential variables u and v, its auxiliary system

$$\begin{aligned} x &= \rho, & u_t &= P^{-2}\rho_x \\ v_x &= P, & v_t &= P^{-2}P_x \end{aligned} \tag{8}$$

admits, as we have found earlier, the infinite-dimensional symmetries:

$$\Gamma_1 = \alpha(t, v)\partial_x - P^2\alpha_v\partial_P - \rho P\alpha_v\partial_\rho, \qquad \Gamma_2 = \alpha(t, v)\partial_u + P\alpha_v\partial_\rho,$$

where $\alpha(t, v)$ is an arbitrary solution of the linear heat equation $\alpha_t = \alpha_{vv}$. These symmetries lead to the local mapping

$$x' = v, \qquad t' = t, \qquad \rho' = \frac{1}{P}, \qquad P' = \frac{\rho}{P}, \qquad u' = x, \qquad v' = u \tag{9}$$

that connects the linear system

$$u'_{x'} = \rho', \qquad u'_{t'} = \rho'_{x'} \qquad v'_{x'} = P', \qquad v'_{t'} = P'_{x'}$$

and the nonlinear system (8). The inverse of (9) leads to the transformation

$$\mathrm{d}x = \rho'\,\mathrm{d}x' + \rho'_{x'}\,\mathrm{d}t', \qquad \mathrm{d}t = \mathrm{d}t', \qquad \rho = \frac{P'}{\rho'}, \qquad P = \frac{1}{\rho'} \tag{10}$$

which connects the linear system

$$\rho'_{t'} = \rho'_{x'x'}, \qquad P'_{t'} = P'_{x'x'}$$

and the nonlinear system (7). Transformation (10) can be written in the integrated form

$$x = \int_{x'_0}^{x'} \rho'\,\mathrm{d}x' + \int_{t'_0}^{t'} (\rho'_{x'})_{x'=x'_0}\,\mathrm{d}t', \qquad t = t' - t'_0, \qquad \rho = \frac{P'}{\rho'}, \qquad P = \frac{1}{\rho'}$$

for some fixed point (x'_0, t'_0).

7. Conclusion

In the present paper, we have classified local first-order conservation laws for a class of systems of nonlinear diffusion equations. This classification has been used to construct inequivalent (with respect to Lie symmetry and equivalence transformations) potential systems for class (2). Namely, we have constructed 32 inequivalent systems that may be used for finding potential symmetries for systems from class (2). Three of them are common for all values of arbitrary elements f and g and the remaining 29 form potential systems for systems (2) with special values of arbitrary elements. We have derived all potential symmetries of (2) associated with 'common' potential systems and to one 'special' potential system. This completes in some sense results of [19] where only partial results on potential symmetries of class (2) were obtained without claiming the completeness of classification, and results of [20] where classification of potential symmetries for class (2) is presented in an implicit form.

Note, that the problem of construction of all possible local (and potential) conservation laws, and therefore, the classification of all possible potential symmetries for class (2) still remains open and can form a subject of future investigation of properties of class (2). Furthermore, the present work is step forward to the classification of potential symmetries for the general system (1).

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